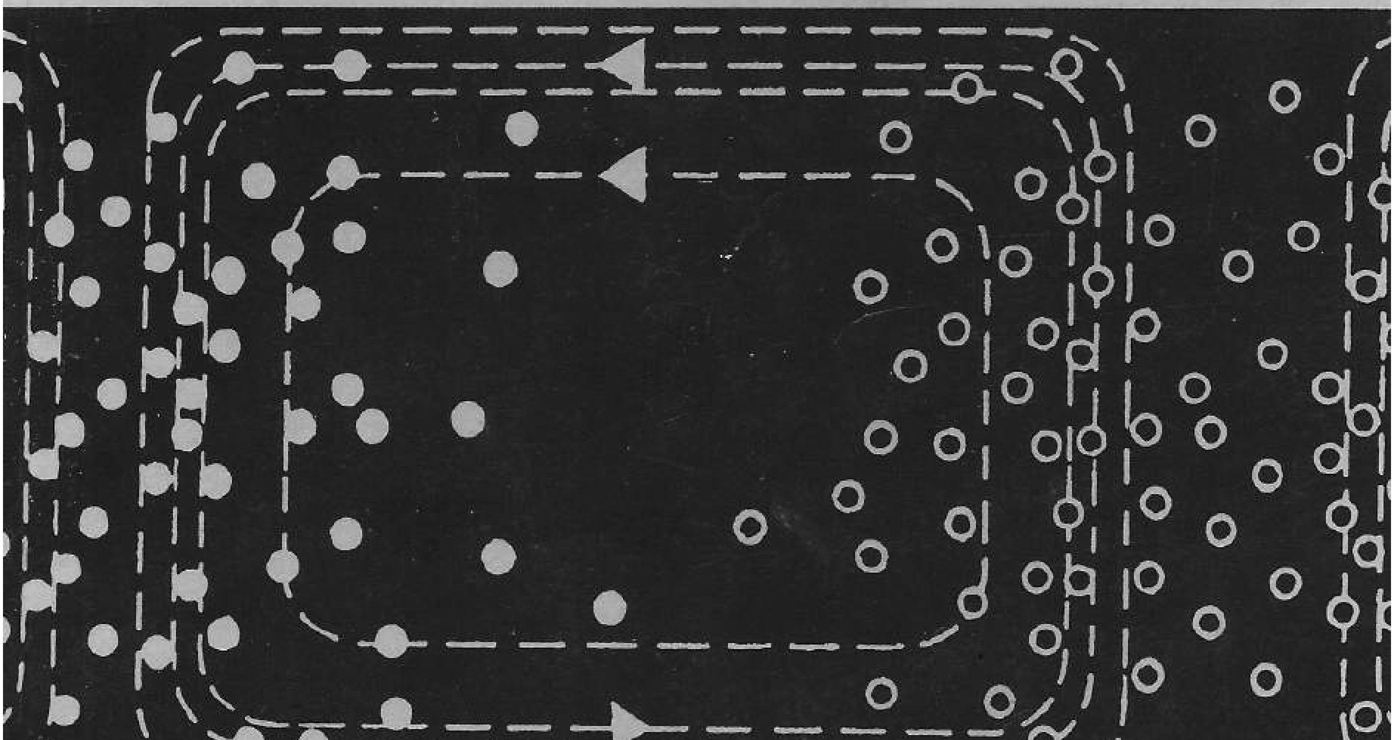
Electricity and Magnetism

Third Edition



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8.3. The Poynting vector of energy flow

Since an electromagnetic wave consists of electric and magnetic fields, we may expect that stored energy is associated with these fields. If the energy density, for which we now use the symbol U, is given by the equations derived for static fields, then for a plane wave in an isotropic dielectric it will be

$$U=\tfrac{1}{2}(\mathbf{D}.\mathbf{E}+\mathbf{B}.\mathbf{H})=\tfrac{1}{2}(\epsilon\mathbf{E}^2+\mu\mathbf{H}^2).$$

For a wave propagated in the positive z-direction, the energy crossing unit area per second will just be the velocity times the energy density. This is

$$vU=\frac{1}{2}\left\{\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}}E_x^2+\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}H_y^2\right\}$$
$$=\tfrac{1}{2}(E_x^2/Z_0+Z_0H_y^2)=E_xH_y=E_x^2/Z_0=Z_0H_y^2.$$

These various relations show that the energy stored in the magnetic field is just equal to that in the electric field. Since the direction of energy flow is normal to both $\mathbf{E}$ and $\mathbf{H}$, and its magnitude is E_xH_y, we can represent the rate of energy flow per unit area by the vector (see Fig. 8.2)

$$\mathbf{E}\wedge\mathbf{H}. \tag{8.23a}$$

This vector is known as the Poynting vector, and it can be seen from eqns (8.21) that the direction of energy flow is reversed for a wave travelling in the opposite direction because the phase of $\mathbf{H}$ relative to $\mathbf{E}$ is reversed.

The value of $\mathbf{E}\wedge\mathbf{H}$ gives the instantaneous rate of energy flow. In a periodic wave the values of $\mathbf{E}$ and $\mathbf{H}$ at any point are oscillating functions of the time, and the mean rate of energy flow is found by averaging $\mathbf{E}\wedge\mathbf{H}$ over a complete period. In general the mean rate of energy flow is (cf. Problem 7.13) per unit area

$$\overline{\mathrm{Re}(\mathbf{E})\wedge\mathrm{Re}(\mathbf{H})}=\tfrac{1}{2}\,\mathrm{Re}(\mathbf{E}\wedge\mathbf{H}^*) \tag{8.23b}$$

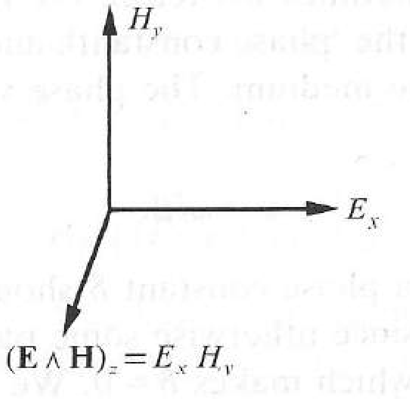

FIG. 8.2. Directions of the electric and magnetic field strengths $\mathbf{E}$ and $\mathbf{H}$ and the Poynting vector $\mathbf{E}\wedge\mathbf{H}$ for a plane wave.

where $\mathbf{H}^*$ is the complex conjugate of $\mathbf{H}$; this expression must be used when the phase difference between $\mathbf{E}$ and $\mathbf{H}$ is not 0 or π.

The Poynting vector is of greater significance than might be thought from the way in which it has been introduced above, and we shall now derive it for the general case. Suppose there is a region of space where an electric field of strength $\mathbf{E}$ causes a current flow of density $\mathbf{J}$. Then the power dissipated per unit volume is $\mathbf{E}.\mathbf{J}$, and the total power dissipated will be

$$\int \mathbf{E}.\mathbf{J}\,\mathrm{d}\tau = \int (\mathbf{E}.\operatorname{curl}\mathbf{H})\,\mathrm{d}\tau - \int \mathbf{E}.(\partial\mathbf{D}/\partial t)\,\mathrm{d}\tau,$$

where we have substituted for $\mathbf{J}$ the expression given by eqn (8.7). Now, using the vector identity

$$\operatorname{div}(\mathbf{E}\wedge\mathbf{H}) = \mathbf{H}.\operatorname{curl}\mathbf{E} - \mathbf{E}.\operatorname{curl}\mathbf{H}$$

and the expression for curl $\mathbf{E}$ given by eqn (8.3), we find that the power dissipated is

$$-\int \mathbf{E}.(\partial\mathbf{D}/\partial t)\,\mathrm{d}\tau - \int \mathbf{H}.(\partial\mathbf{B}/\partial t)\,\mathrm{d}\tau - \int \operatorname{div}(\mathbf{E}\wedge\mathbf{H})\,\mathrm{d}\tau.$$

The first two terms may be written as

$$-\int\left(\mathbf{E}\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial\mathbf{B}}{\partial t}\right)\mathrm{d}\tau,$$

and this is the rate at which the stored energy increases, since the work done in infinitesimal changes $\mathrm{d}\mathbf{D}$, $\mathrm{d}\mathbf{B}$ at constant $\mathbf{E}$, $\mathbf{H}$ is

$$\int (\mathbf{E}\,\mathrm{d}\mathbf{D} + \mathbf{H}\,\mathrm{d}\mathbf{B})\,\mathrm{d}\tau.$$

If ϵ, μ are independent of $\mathbf{E}$, $\mathbf{H}$ and of time, the first two terms may be written as

$$-\frac{1}{2}\frac{\partial}{\partial t}\int (\mathbf{D}.\mathbf{E} + \mathbf{B}.\mathbf{H})\,\mathrm{d}\tau,$$

which we may interpret as the rate at which the energy stored in the electromagnetic field diminishes. The last term may be transformed using eqn (A.8) of Appendix A into the surface integral

$$-\int (\mathbf{E}\wedge\mathbf{H})\,.\,\mathrm{d}\mathbf{S}$$

taken over the surface bounding the volume under consideration. It represents the rate at which energy flows into the volume under consideration, and we may interpret the vector $\mathbf{E}\wedge\mathbf{H}$ as the rate at which energy flows across unit area of the boundary. Strictly speaking, only the integral